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Applicability of an Asymptotic Expansion for Elastic Buckling Problems with Mode Interaction

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Asymptotic expansions often provide relatively simple formulas for post-buckling and imperfection sensitivity analyses. In a previous work, Byskov and Hutchinson developed and utilized an expansion to determine the increase in imperfection sensitivity caused by interaction between two or more buckling modes. This article contains an attempt to assess the range of validity of that expansion. The estimates of the applicability are made with reference to a simple example.

Nomenclature

a_{ijk} = postbuckling coefficient, see Eq. (7)
A = cross-sectional area of a longeron
b_{ijkm} = postbuckling coefficient, see Eq. (7)
D_I = see Eqs. (A3) and (A5)
E = Young's modulus
f_i = amplitude of w_i
H = distance between longerons
I = moment of inertia of a longeron
K_{ijl} = see Eqs. (A3) and (A4)
ℓ = bay length of truss
L = length of truss column
M = number of interacting modes
n = L/ℓ = number of bays
N_I^α = normal force in longeron number α from overall buckling
N_α = N_α^α = normal force in longeron number α from local mode number α
P_2 = local mode critical load
u, u_0, u_i, u_{ij} = see Eq. (6)
w_i = transverse displacement component of buckling mode number i
x = coordinate
α = local mode number range 2, 3
ζ = x/L
η_i = linear combinations of ξ_i, see Eq. (A8)
$\bar{\eta}_i$ = imperfections corresponding to η_i
λ = scalar load parameter
λ_B = value of λ at bifurcation for locally imperfect structure
λ_s = maximum of λ for imperfect structure
λ_I = value of λ at overall mode bifurcation
λ_2 = value of λ at local mode bifurcation
ξ_i = dimensionless amplitude of mode number i
$\bar{\xi}_i$ = imperfection amplitude corresponding to mode number i
Π_p = potential energy of structure

I. Introduction

IN a series of articles,[1,2,3] Koiter and his coworkers have developed a method based on the concept of a "slowly varying" local mode amplitude. This method can, to a high degree of accuracy, describe the nonlinear interaction between overall and local buckling modes in elastic structures. The major effect of this interaction usually lies in increased imperfection sensitivity, compared with the predictions from a one-mode analysis. Although the foundation of the method can be interpreted in a straightforward physical way, its derivation is fairly sophisticated and based on the assumption that the wavelength of the overall mode is many times that of the local mode. The method established by Byskov and Hutchinson[4] involves manipulations of a more standard type and requires no assumption concerning the wavelengths of the two modes, although the expansion may cease to be valid when the wavelength of the overall mode is very large compared to the wavelength of the local mode. This article addresses itself to some other aspects of the problem of the validity range of the latter method. The estimates are made on the basis of a simple example akin to one investigated by Thompson and Hunt[5] and by Crawford and Hedgepeth.[6] It is shown that the ratio between the overall mode buckling load and the local mode buckling load constitutes one of the basic parameters determining the range of validity.

II. Structural Problems

A. Truss Column

The truss column, see Fig. 1, is of a linearly elastic material with Young's modulus E. It is taken to be long and slender, in the sense that its number of bays $n \gg 1$, that its length-height ratio $L/H \gg 1$, and that end-effects can be neglected. In the analysis, we take into account only the stiffness coming from the longerons whose moment of inertia is I and cross-sectional area is A. We will also assume that the longerons, by themselves, are slender, such that $n^2 I/(AL^2) \ll 1$. Furthermore, we will consider deformations in the plane of the column only.

In contrast to what was done in Refs. 5 and 6, we take the longerons to be continuous over all bays. If we neglect this continuity there is no possibility of redistribution of forces. In that way an essential feature would be lost, and the two methods of analysis applied below would only duplicate the results from Refs. 5 and 6.

The modes that may interact in this built-up column are an overall Euler mode $u_I(x)$ with wavelength L and two local Euler modes $u_2(x)$ and $u_3(x)$ both with wavelength L/n, where $u_i(x)$, $i = 1,2,3$ designates all components of the displacement field for mode number i, and where x is the coordinate along the column. The two local modes do not

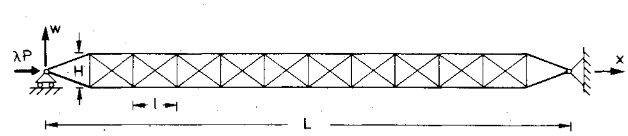

Fig. 1 Truss column.

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interact because they are confined to different structural components.

To make things clear, we limit ourselves to cases for which the only imperfections present are local sinusoidal imperfections of equal amplitude in all bays.

The overall mode, lower index 1, gives rise to transverse displacements $w_I(x)$ of the column axis,

$$w_I(x) = f_I \sin[\pi(x/L)] \tag{1}$$

and normal forces in the flanges,

$$N_I^\alpha(x) = f_I(-1)^\alpha \tfrac{1}{2} EAH(\pi/L)^2 \sin[\pi(x/L)] \tag{2}$$

where $\alpha = 2,3$ corresponds to the upper and the lower longeron, respectively, and f_I designates the amplitude of the mode.

The local modes, lower index $\alpha = 2,3$, produce no normal forces in the longerons, only transverse displacements:

$$w_\alpha(x) = f_\alpha \sin[n\pi(x/L)] \tag{3}$$

$$N_\alpha(x) = N_\alpha^\alpha(x) = 0 \tag{4}$$

If there are no imperfections, then the column will buckle at the smaller of the critical overall load factor λ_I and the critical local load factor λ_2. In the following, we will normalize the loads such that $\lambda_2 = 1$.

In the presence of symmetric local imperfections, the transverse displacements in the flanges will grow with increasing load. The column axis, however, will remain straight until the load factor λ has reached a value $\lambda_B < \min(1, \lambda_I)$, at which the column bifurcates into some curved shape. The method of Refs. 1 and 3 is very well suited to closely describing the above process, and its results in terms of λ_B may therefore be taken to be correct. After a fairly cumbersome analysis (see Appendix), we get the following expression:

$$(\lambda_I - \lambda_B)(1 - \lambda_B)^3 = \tfrac{1}{2}\lambda_B^3 \bar{\eta}_2^2 \tag{5}$$

where $\bar{\eta}_2 = (\bar{\xi}_2 - \bar{\xi}_3)/2$, $\bar{\xi}_\alpha$ being the local imperfection amplitudes.

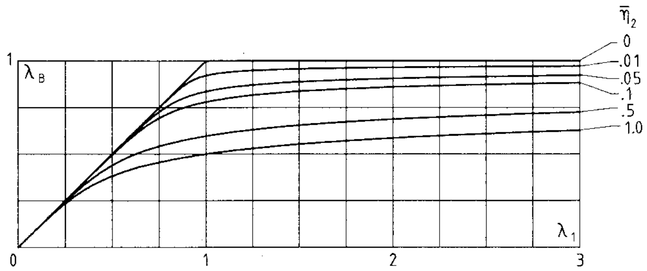

Fig. 2 Carrying capacity of truss column obtained by the slowly varying amplitude method.

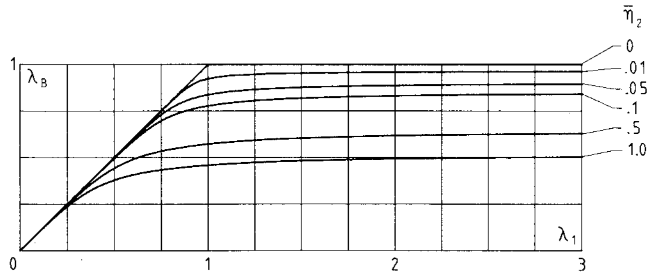

Fig. 3 Carrying capacity of truss column obtained by the asymptotic expansion method.

For a perfect structure with a linear prebuckling state whose displacement field is denoted by λu_0 the asymptotic expansion from Ref. 4 assumes the total displacement field to be

$$u = \lambda u_0 + \xi_i u_i + \xi_i \xi_j u_{ij} + \ldots \tag{6}$$

where the range of the indices is $[1, M]$, M being the number of participating modes, and where a repeated lower-case index denotes the summation from 1 to M. In Eq. (6), ξ_i is the amplitude of buckling mode number i. Utilizing Eq. (6) and similar expansions for the strain and stress fields, the method of Ref. 4 yields the following set of nonlinear equations for the determination of the snap buckling load λ_S or the bifurcation load λ_B for an imperfect structure:

$$(1 - \lambda/\lambda_I)\xi_I + \xi_i \xi_j a_{ijI} + \xi_i \xi_j \xi_k b_{ijkI}$$

$$= (\lambda/\lambda_I)\bar{\xi}_I, \quad I = 1, \ldots, M \tag{7}$$

(no sum on I). In Eq. (7), λ_I denotes the critical load factor corresponding to mode I and $\bar{\xi}_I$ denotes the imperfection amplitude in direction of buckling mode number I. The reader is referred to Ref. 4 for the complete formulas for the coefficients a_{ijI} and b_{ijkI}.

In the present problem it may be shown that the coefficients b_{ijkI} have no influence on Eq. (8) below, and therefore we do not need the second-order fields u_{ij}. In the Appendix we find the expression corresponding to Eq. (5):

$$(\lambda_I - \lambda_B)(1 - \lambda_B)^3 = (4/\pi^2)\lambda_I \lambda_B^2 \bar{\eta}_2^2 \tag{8}$$

As will be seen from Figs. 2 and 3, the numerical values determined from Eqs. (5) and (8) differ little for imperfections $\bar{\eta}_2 < 1$ ($\bar{\eta}_2 = 1$ means that the local mode imperfection amplitude is equal to the radius of gyration of the longerons) as long as $\lambda_I <$ appr. 1. For $\lambda_I \to \infty$, an essential difference is observed in that Eq. (5) gives $\lambda_B \to 1$ irrespective of the imperfection level, whereas Eq. (8) suggests that for small imperfections

$$\lambda_B \to 1 - [(4/\pi^2)\bar{\eta}_2^2]^{1/3} \tag{9}$$

The reason for this obviously wrong result from Eq. (8) may be understood after inspection of Figs. 4 and 5:

Both curves illustrate the relation

$$\eta_2 = [\lambda/(1 - \lambda)]\bar{\eta}_2 \tag{10}$$

which together with $\eta_I = \xi_I = 0$ and $\eta_3 = (\xi_2 + \xi_3)/2 = 0$ determines the fundamental path in the (λ, η_i) space. In both figures, A denotes the point at which bifurcation away from $\eta_I = 0$ takes place. When $\lambda_I < 1$, point A lies in the immediate neighborhood of the point $(\lambda, \eta_i) = (\lambda_I, 0, [\lambda_I/(1 - \lambda_I)]\bar{\eta}_2, 0)$, which for decreasing imperfection level moves closer to the point $(\lambda_I, 0, 0, 0)$ about which we do the expansion, Eq. (6). This is not the case when $\lambda_I > 1$ because then $\lambda_B \to 1$ as λ_I increases, which in its turn implies that A tends to the point $(1, 0, \infty, 0)$. Due to the fact that this latter point lies far away from the origin of the expansion we must expect this to be of questionable validity for large values of λ_I, as is demonstrated by Eq. (9).

Since $\lambda_I > \lambda_B$, it may be observed by comparing Eq. (8) with Eq. (5) that the first overestimates the imperfection sensitivity for large values of the imperfections. However, even at the unrealistically high level $\bar{\eta}_2 = 0.5$, the difference between the results does not exceed 4% for $\lambda_I < 1$, and for $\bar{\eta}_2 = 0.1$ it is less than 1%.

The above-described kind of loss of accuracy of results obtained by the asymptotic expansion must be anticipated for structures with local modes that are postbuckling stable or neutral, as in the present problem, provided the design is such that $\lambda_I > 1$.

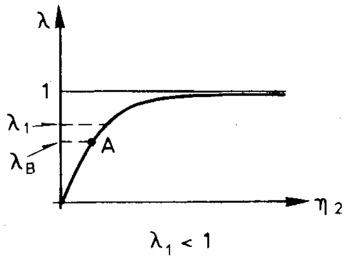

Fig. 4 Load-displacement relation for locally imperfect truss column with small overall buckling load.

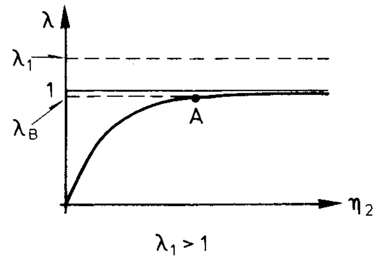

Fig. 5 Load-displacement relation for locally imperfect truss column with large overall buckling load.

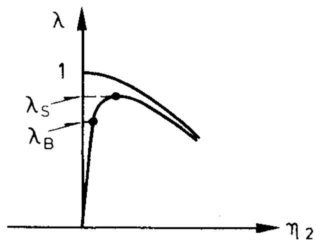

Fig. 6 Load-displacement relation for a locally imperfect structure with a postbuckling unstable local mode.

B. Axially Stiffened Cylindrical Shells

The mode interaction problem for these structures has been the subject of extensive analyses in Refs. 3 and 4, and the reader is referred to these works for complete details.

The local mode may be postbuckling stable, neutral or unstable, depending on the distance between stringers. When the first is the case, Koiter's method predicts that for increasing imperfection level, λ_B may approach an asymptotic value greater than zero. This behavior is not described by the results from Ref. 4, which will yield pessimistic values of λ_B. If, however, the local mode is postbuckling unstable, we can get almost identical values of λ_B (Ref. 7) even for designs with $\lambda_1 > 1$. The reason for this can be understood by examining Fig. 6.

Bifurcation into the overall mode must take place at a lower load level than λ_s, which in this context designates the snap buckling load determined from a local one-mode analysis. As long as λ_s may be regarded as correct, the same will hold for λ_B because the point corresponding to λ_B lies even closer to the fundamental state for the perfect structure than does the snap buckling point.

III. Discussion

Within the scope of the assumptions, Koiter's method gives correct results for structures with interaction between short-wave local modes and long-wave overall modes. Its application requires a thorough understanding of the behavior of the structure in addition to a fair amount of mathematical skill.

In principle, the asymptotic expansion of Byskov and Hutchinson is readily applied since it generates a sequence of linear problems for u_0, u_i and u_{ij}. On the other hand, its range of validity for small imperfection levels is limited, in that either the design has to be such that the local mode critical load is greater than the overall, or the local mode must, in itself, be postbuckling unstable. The first condition is met in almost all practical structural designs, and thus the second condition is often unimportant. For structures with a postbuckling stable local mode, Koiter's method may predict a lower bound on the carrying capacity for increasing local imperfections. The asymptotic expansion method is incapable of describing this feature, which is not discussed in the body of the paper where only small initial imperfections are considered. Finally, it may be worthwhile mentioning that the method from Ref. 4 is easily formulated in terms of finite elements.

Appendix

The Byskov-Hutchinson Method

The prebuckling state for the perfect structure consists of a uniform compression of the longerons. The normal forces in this state are

$$N_0^\alpha = -\tfrac{1}{2}\lambda P_2 \tag{A1}$$

with

$$P_2 = \pi^2 (EI/\ell^2) = (n\pi)^2 (EI/L^2) \tag{A2}$$

where EI is the bending stiffness of the longerons.

For the present problem Eq. (5) of Ref. 4 may immediately give

$$a_{ijI} = K_{ijI}/D_I \tag{A3}$$

with

$$K_{ijI} = \sum_{\alpha=2}^{3} \int_0^L (N_I^\alpha w_{i,x}^\alpha w_{j,x}^\alpha + 2N_i^\alpha w_{j,x}^\alpha, w_{I,x}^\alpha)\,\mathrm{d}x \tag{A4}$$

and

$$D_I = \lambda_I \sum_{\alpha=2}^{3} \int_0^L P_2 (w_{I,x}^\alpha)^2 \mathrm{d}x \tag{A5}$$

where summation does not apply to repeated upper-case indices.

The only nonvanishing N_I^α are N_I^α with the result that

$$a_{122} = a_{212} = -a_{133} = -a_{313} = \frac{2}{\pi}\lambda_I \tag{A6a}$$

$$a_{221} = -a_{331} = \frac{1}{4\pi}\frac{1}{\lambda_I} \tag{A6b}$$

$$\text{all other } a_{ijk} = 0 \tag{A6c}$$

where we have taken

$$f_I = H \text{ and } f_\alpha = \sqrt{I/A} \tag{A7}$$

After introduction of

$$\eta_I = \xi_I, \quad \eta_2 = \tfrac{1}{2}(\xi_2 - \xi_3), \quad \eta_3 = \tfrac{1}{2}(\xi_2 + \xi_3) \tag{A8}$$

and like expressions for $\bar{\eta}_i$ with only $\bar{\eta}_2 \neq 0$ Eqs. (7) may be written

$$(\lambda_I - \lambda)\eta_I + (1/\pi)\eta_2\eta_3 = 0 \tag{A9}$$

$$(1-\lambda)\eta_2 + (4/\pi)\lambda_I\eta_I\eta_3 = \lambda\bar{\eta}_2 \tag{A10}$$

$$(1-\lambda)\eta_3 + (4/\pi)\lambda_I\eta_I\eta_2 = 0 \tag{A11}$$

The prebifurcation solution is clearly

$$\eta_1 = \eta_3 = 0, \quad \eta_2 = [\lambda/(1-\lambda)]\bar{\eta}_2 \tag{A12}$$

On the bifurcated path we have $\eta_1 \neq 0$ and/or $\eta_3 \neq 0$, and Eqs. (A9) and (A11) give

$$(\lambda_1 - \lambda) - \frac{4}{\pi^2}\frac{\lambda_1}{1-\lambda}\eta_2^2 = 0 \tag{A13}$$

When we equate η_2 from Eq. (A12) with η_2 from Eq. (A13), the expression Eq. (8) is found.

Koiter's Method

The crux of this method lies in the assumption that the local mode amplitudes η_α may vary slowly along the built-up column. We may conveniently carry out the analysis with Ref. 1 as basis. It turns out that, before bifurcation, η_2 does not vary along the column, but that the post-bifurcation local amplitude $\delta\eta_2$ indeed does so.

The following notations are introduced in order to simplify the expression for the potential energy:

$$\zeta = x/L \tag{A14}$$

$$(\dot{\ }) = \mathrm{d}(\ )/\mathrm{d}\zeta \tag{A15}$$

$$w_1(\zeta) = \eta_1(\zeta)H \tag{A16}$$

$$w_2(\zeta) = [\eta_2(\zeta) + \eta_3(\zeta)]\sin(n\pi\zeta)\sqrt{I/A} \tag{A17}$$

$$w_3(\zeta) = [\eta_3(\zeta) - \eta_2(\zeta)]\sin(n\pi\zeta)\sqrt{I/A} \tag{A18}$$

It may be observed that the meaning of η_i has been changed slightly, such that η_1 now denotes the dimensionless overall mode and not just the amplitude and such that the local mode amplitude combinations η_α may vary along the column. Thus, we have employed our advance knowledge of the sinusoidal shape of the local modes, whereas we do not have to do so as regards the overall mode.

After some manipulation, we may get the potential energy Π_p for this structure:

$$\Pi_p = \Pi_P^0 + ELn^4\left(\frac{\pi}{L}\right)^4 I^2 \int_0^1 \left(\frac{16}{\pi^4}\lambda_1[\lambda_1\ddot{\eta}_1^2 - \pi^2\lambda\dot{\eta}_1^2] + 2(1-\lambda)(\eta_2^2+\eta_3^2) + \frac{8}{\pi^2}\lambda_1\ddot{\eta}_1\eta_2\eta_3 + \eta_2^2\eta_3^2 - 4\lambda\bar{\eta}_2\eta_2\right)\mathrm{d}\zeta \tag{A19}$$

where Π_P^0 denotes terms of order zero in the buckling modes.

The first variation of Π_P provides us with Eqs. (A12) for the prebifurcation path.

After some work, the second variation of Π_p furnishes us with expressions for the postbifurcation fields $\delta\eta_i$ and finally with Eq. (5). The analysis leading to Eq. (5) will show that $\delta\eta_\alpha$, as well as $\delta\eta_1$, vary as $\sin(\pi\zeta)$.

The fact that η_2 does not depend on ζ and that $\delta\eta_i$ do, cannot be described by the asymptotic expansion, Eq. (6), which, in a sense, assumes that η_i and $\delta\eta_i$ are of the same shape.

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